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TWO-DIMENSIONAL GRAINED COMPOSITES OF MINIMUM STRESS CONCENTRATION

SHMUEL VIGDERGAUZ
Research and Development Division, Israel Electric Corporation Ltd., P.O.B. 10, Haifa 31000, Israel

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Abstract—In this paper, we consider a regular two-phase microstructure which is formed by identical foreign inclusions spaced periodically in an elastic matrix. The optimization problem of finding the inclusion shape that minimizes the stress concentration over the whole region is solved explicitly (without recourse to approximate or numerical methods) when both materials have the same shear modulus differing only in Poisson's ratios. It is rigorously shown that the stress concentration reduces to the absolute minimum (dictated only by a global part of a given average stress tensor) if the inclusions have equi-stress boundaries identified before in the closely-related context of energy-wise optimization. The stress concentration factors at non-optimal circular inclusions are also found in a closed form to estimate the gain from applying optimal contours.

1. INTRODUCTION

The design of grained composites is often formulated as an optimization problem with inclusion shapes used as designing variables. Whereas a large amount of theoretical research has been performed concerning the extremal effective elastic moduli, coefficient of thermal expansion, stress–strain energy and other average (integral) characteristics of composites, minimization of their local stress levels have been investigated to a much lesser extent. The problem we are interested in is to find the optimum shape of elastic inclusion which minimizes the stress concentration throughout the structure with the principal cell T subjected to an average stress field σ_0

$$\max_{z \in T} \|\sigma(z)\| \rightarrow \min_{\{L\}} = v_{min} \tag{1}$$

where $\{L\}$ denotes the admissible set of smooth closed curves bounding a given area and $\|\cdot\|$ stands for some local functional over the stress tensor $\sigma(z)$. Mathematically, local objective functionals are harder to handle because even the occurrence of their extrema is not known in advance. The first to be investigated was the simplest case of a finite number of holes or rigid inclusions in an elastic plate (Banichuk, 1974; Vigdergauz, 1974) when the traction components are predetermined on the interface. The now well-studied equal-stress contours were found to be optimal for the most-used deviatoric criterion (Cherepanov, 1979)

$$\|\sigma(z)\| \equiv |\sigma(z) - \tfrac{1}{2}\mathbf{Tr}\sigma(z)|^2 = v(z) \tag{2}$$

with its maximum occurring at each boundary point. As for *elastic* inclusions, the only analytic work which has been done so far is that by Grabovsky and Kohn (1995). Starting from the argument of Wheeler (1994), these authors proved the equi-stress curves are likewise optimal for the specific norm

$$\|\sigma(z)\| = \sup|\sigma v|; \quad |v| = 1. \tag{3}$$

Contrary to (2), this norm is *linear* in components of $\sigma(z)$.

The objective of the present work is to prove the optimality of the equi-stress inclusions for the less simple *quadratic* criterion (2). Previously these inclusions were identified as an optimal solution which minimizes the cell elastic energy (Vigdergauz, 1986 and 1994; Grabovsky and Kohn, 1995). Generally, while dealing with grained composites, it is desirable to separate the material constitutive equations linked to the interface through certain contact conditions. In the works cited above it was done by assuming that the phase interaction includes only uniform normal stress at the boundary sought. Here we restrict ourselves to the specific case when both phases are identical in the shear modulus differing only in Poisson's ratios. It has the same effect of simplifying the initial boundary conditions. Severe as this limitation might seem, there is still a meaningful problem to be solved. Otherwise, the approach used is much like the one suggested by the author (1996). The following is the outline of the paper. In Section 2 we fix notations and pose the standard doubly periodical boundary-value problem of elasticity using the potentials of Kolosov–Muskhelishvili. In Section 3 the complex variable technique is employed to derive some preliminary identities for later use. In Section 4 we obtain the necessary condition of optimality which results in the same inverse problem for equi-stress boundaries. As before (Vigdergauz, 1994; Grabovsky and Kohn, 1995), this optimal solution allows any external field σ_0 so long as its deviatoric part remains sufficiently small [see inequality (45)]. The minimal value of the stress concentration factor (2) is also found explicitly as a rational function of the given data. For purposes of comparison, we analytically solve the direct problem of elasticity in the cell with a circular inclusion. It is done in Section 5 by using the Weierstrass elliptic functions. The result obtained is valid for any geometrically admissible volume fraction. Finally, some comments regarding the practically applicability of the work are offered in Section 6.

2. BASIC NOTATIONS AND CELL FORMULATION

Consider the setup described in Fig. 1. Locate the composite plate in the plane of a complex variable $z = x + iy$. Let the sides of rectangular cells be parallel to the axes of the Cartesian coordinate system XOY and the fundamental cell T be centered at its origin. Also, let the index $j = 1, 2$ denote the inclusion and the matrix, respectively. We adopt the following notations: S_j is the part of T occupied by the corresponding material, h_j is the area of S_j, H is the cell area, $c_j = S_j/H$ is the phase volume fraction, $c_1 + c_2 = 1$, L is the smooth inclusion boundary, and, finally, t is an arbitrary point on L.

The primitive lattice periods are usually symbolized (Abramowitz and Stegun, 1965) by $2\omega_1 = \overline{EF}$ and $2\omega_2 = \overline{GH}$. The following evident relations can be also written out for later use:

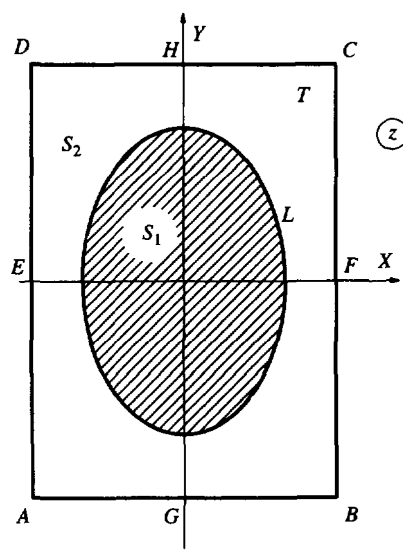

Fig. 1. The schematic picture of two phase regular structure.

$$H = -4i\omega_1\omega_2 \tag{4}$$

and [see Muskhelishvili, (1977)]:

$$\int_L \bar{t}\,dt = i\int_L (x\,dy - y\,dx) = -2ih_1 \tag{5}$$

where the integration is performed in a clockwise direction.

Suppose that a doubly periodic stress field $\sigma(z)$ has fixed average values σ_{xx}^0, σ_{yy}^0 and $\tau_{xy}^0 = 0$ which represent the constant tensor σ_0:

$$\mathbf{Tr}\sigma_0 = \sigma_{xx}^0 + \sigma_{yy}^0 = \frac{1}{H}\int_{S_1} \mathbf{Tr}\sigma(z)\,dx\,dy + \frac{1}{H}\int_{S_2} \mathbf{Tr}\,\sigma(z)\,dx\,dy \tag{6}$$

$$\mathbf{Dev}\sigma_0 = \sigma_{yy}^0 - \sigma_{xx}^0 = \frac{1}{H}\int_{S_1} \mathbf{Dev}\sigma(z)\,dx\,dy + \frac{1}{H}\int_{S_2} \mathbf{Dev}\,\sigma(z)\,dx\,dy. \tag{7}$$

The elastic field of the cell can be described at any point $z \in T$ using the Kolosov–Muskhelishvili complex potentials $\varphi_j(z)$, $\psi_j(z)$, which are analytic, respectively, in the subdomains S_j, $j = 1, 2$ (Muskhelishvili, 1975). These functions are related to the displacement and stress components in T by

$$2\mu_j[u_x(z) + iu_y(z)] = \kappa_j\varphi_j(z) - z\overline{\varphi_j'(z)} - \overline{\psi_j(z)} \tag{8}$$

$$\sigma_{xx}(z) + \sigma_{yy}(z) = \sigma_{nn}(z) + \sigma_{\tau\tau}(z) = 4\,\mathrm{Re}\,[\varphi_j'(z)] \tag{9}$$

$$\begin{aligned}\sigma_{yy}(z) - \sigma_{xx}(z) - 2i\sigma_{xy}(z) &= (n_x - in_y)^2[\sigma_{nn}(z) - \sigma_{\tau\tau}(z) - 2i\sigma_{n\tau}(z)] \\ &= 2[\bar{z}\varphi_j''(z) + \psi_j'(z)]; \quad z \in S_j\end{aligned} \tag{10}$$

where $\kappa_j = (3 - \nu_j)/(1 + \nu_j) > 1$, and ν_j, μ_j are the elastic moduli of the materials, while $n(z) = \{n_x, n_y\}$ and $\tau(z) = \{\tau_x, \tau_y\}$ stand for the basic vectors of a local system of curvilinear orthogonal coordinates. A prime indicates differentiation with respect to z, and a bar indicates complex conjugation.

Assuming perfect bonding at the matric–inclusion interface, the potentials are linked to L through the following conditions (Muskhelishvili, 1975):

$$\varphi_2(t) + t\overline{\varphi_2'(t)} + \overline{\psi_2(t)} = \varphi_1(t) + t\overline{\varphi_1'(t)} + \overline{\psi_1(t)} = f(t); \quad t \in L \tag{11}$$

$$\mu_1(\kappa_2\varphi_2(t) - t\overline{\varphi_2'(t)} - \overline{\psi_2(t)}) = \mu_2(\kappa_1\varphi_1(t) - t\overline{\varphi_1'(t)} - \overline{\psi_1(t)}). \tag{12}$$

Complex-valued function $f(t)$ is given by the integral

$$f(t) = \int_{t_0}^{t} (\sigma_{nn}(\xi) + i\sigma_{n\tau}(\xi))\,d\xi. \tag{13}$$

Combining eqn (11) with eqn (12) we also obtain

$$\mu_2(\kappa_1 + 1)\varphi_1(t) = \mu_2(\kappa_1 + 1)\varphi_1(t) + (\mu_2 - \mu_1)f(t). \tag{14}$$

Let $I_1(z)$, $I_2(z)$ be the stress tensor invariants:

$$I_1(z) = \sigma_{nn}(z)+\sigma_{\tau\tau}(z) = 4\,\mathrm{Re}\,\varphi_j'(z) \tag{15}$$

$$I_2(z) = \sigma_{n\tau}^2(z)-\sigma_{nn}(z)\sigma_{\tau\tau}(z); \quad z\in S_j; \quad j=1,2. \tag{16}$$

By (9), (10), (15) and (16), the deviatoric criterion $v(z)$ from (2) is then written at each point of the plate as follows

$$v(z) = |\bar{z}\varphi_j''+\psi_j'(z)|^2; \quad z\in S_j, \quad j=1,2. \tag{17}$$

The curve L is a line of discontinuity of $v(z)$. For functions of this kind, their limit values on L from S_1 and S_2 are denoted by the superscripts plus and minus, respectively.

The stress field periodicity results in the specific translation properties of the potentials $\varphi_2(z)$, $\psi_2(z)$ (Muskhelishvili, 1975). In particular, the function $\varphi_2'(z)$ is doubly periodic, hence $\varphi_2(z)$ is quasi-periodic:

$$[\varphi_2(z)]_j = 2A_j\omega_j; \quad [\varphi_2'(z)]_j = 0; \quad j=1,2. \tag{18}$$

Here A_1, A_2 are the (real) constants to be found later and the square brackets denote the differences of the bracketed function values at arbitrary congruent points: $[z]_j = 2\omega_j, j = 1, 2$.

Next, integrating the second identity (10) we have, in view of (17):

$$[\psi_2(z)]_1 = \mathrm{Const}_1-2\varphi_2'(z)\omega_1; \quad [\psi_2(z)]_2 = \mathrm{Const}_2+2\varphi_2'(z)\omega_2. \tag{19}$$

For the potentials $\varphi_1(z)$, $\psi_1(z)$, contrastingly, no relations of periodicity exist, since these functions are defined in a simply connected region.

3. GENERAL RELATION

The original problem of finding the stresses in T can thus be replaced by the problem of finding the four analytic functions φ_j and ψ_j, $j = 1, 2$ which satisfy (11) and (12), and (18) and (19).

However, the latter problem is still hard to solve exactly due to the complexity of the boundary conditions (11) and (12). To simplify the matter we restrict our study to the case of $\mu_1 = \mu_2$ when the pairs $(\varphi_1(z), \varphi_2(z))$ and $(\psi_1(z), \psi_2(z))$ of the unknown functions can be found separately.

Here, eqn (14) becomes

$$(\kappa_1+1)\varphi_1(t) = (\kappa_2+1)\varphi_2(t). \tag{20}$$

By the principle of analytic continuation (Alhfors, 1965), identity (20) permits a new quasi-periodic, holomorphic function $\Omega(z)$ to be defined in T as follows

$$\Omega(z) = (\kappa_j+1)\varphi_j(z); \quad z\in S_j+L; \quad j=1,2.$$

Since $\Omega(z)$ has no singularities in the whole z-plane, it reduces to a linear function (an immediate consequence of Liouville's theorem. See Alhfors, 1965)

$$\Omega(z) = a_0z+b; \quad z\in T.$$

The arbitrary constant b can be neglected in what follows. Hence, the functions $\varphi_1(z)$, $\varphi_2(z)$ are

$$\varphi_j(z) = \frac{a_0}{\kappa_j+1}z; \quad z \in S_j; \quad j = 1,2$$

$$A_1 = A_2 = \frac{a_0}{\kappa_2+1}. \tag{21}$$

Then we conclude from (15) that the invariant $I_1(z)$ takes individual constant values in the regions of homogeneity S_1, S_2 of the cell

$$I_1(z) = \frac{4a_0}{(\kappa_j+1)}; \quad z \in S_j + L; \quad j = 1,2 \tag{22}$$

and, consequently,

$$I_1^+(z) - I_1^-(z) = 2d_0 = \frac{4a_0(\kappa_2-\kappa_1)}{(\kappa_1+1)(\kappa_2+1)}. \tag{23}$$

Substituting (21) into (22) yields, after some algebra,

$$4a_0 = \frac{(\kappa_1+1)(\kappa_2+1)}{c_1\kappa_2+c_2\kappa_1+1}\mathbf{Tr}\sigma_0. \tag{24}$$

With (21) we next eliminate the functions $\varphi_j(z)$, $\varphi_j'(z), j = 1,2$ from either of two eqns (11), (12) to obtain, after conjugating,

$$\psi_2(t) - \psi_1(t) = d_0\bar{t}. \tag{25}$$

Similarly, conditions (19) possess the form

$$[\psi_2(z)]_1 = b_1; \quad [\psi_2(z)]_2 = b_2 \tag{26}$$

which is to say that for *any* contour L, $\psi_2(z)$ is quasi-periodic. Because of the symmetry about the x- and y-axes the constants b_1, b_2 are real and pure imaginary, respectively. To define them, we use analyticity of the potential $\psi_2(z)$ much as was done by the author (1996) for the more readily solved energy-wise problem. To this end let us integrate $\psi_2(z)$ over the boundary of the region S_2 in which this function has no singularities. Therefore, the integral vanishes (Alhfors, 1965)

$$\int_A^B \psi_2(z)\,\mathrm{d}z + \int_B^C \psi_2(z)\,\mathrm{d}z + \int_C^D \psi_2(z)\,\mathrm{d}z + \int_D^A \psi_2(z)\,\mathrm{d}z + \int_L \psi_2(t)\,\mathrm{d}t = 0.$$

By (25) and (5), the last item becomes

$$\int_L \psi_2(t)\,\mathrm{d}t = \int_L \psi_1(t)\,\mathrm{d}t + d_0\int_L \bar{t}\,\mathrm{d}t = -2\mathrm{i}d_0h_1 \tag{27}$$

for the function $\psi_1(z)$ is holomorphic in (S_1+L) and, similarly, makes no contribution to the curvilinear integral over L.

In the third and fourth integral, write $z+\omega_2$, $z+\omega_1$, respectively, for z:

$$\int_B^C [\psi_2(z)]_1\,\mathrm{d}z - \int_A^B [\psi_2(z)]_2\,\mathrm{d}z = 2\mathrm{i}d_0h_1.$$

According to (26), the integrands in the foregoing identity are constant. Thus, we have

$$b_1\omega_2 - b_2\omega_1 = \mathrm{i}d_0 h_1. \tag{28}$$

In a like manner, the second equation in b_1, b_2 may be deduced from the identity (7). By applying Cauchy–Riemann relations (Alhfors, 1965) to the harmonic functions $P_j(x, y) = \mathrm{Re}\,\psi_j(z)$ and $Q_j(x, y) = \mathrm{Im}\,\psi_j(z), j = 1, 2$, it results from (10) and (21) that

$$\frac{\partial P_j}{\partial x} + \frac{\partial Q_j}{\partial y} = \sigma_{yy}(z) - \sigma_{xx}(z)\,; \quad z \in S_j, \quad j = 1, 2. \tag{29}$$

We next substitute (29) in (7) and use Stokes formula (Mikhlin, 1965) to transform the integrals over S_j into those taken along the boundaries ∂T and L. Since $\psi_1(z)$ and $\psi_2(z)$ are linked on L by (25) we arrive at

$$\frac{1}{H}\int_{\partial T} P_2\,\mathrm{d}y - Q_2\,\mathrm{d}x + \frac{d_0}{H}\int_L x\,\mathrm{d}y - y\,\mathrm{d}x = \sigma^0_{yy} - \sigma^0_{xx}.$$

Unlike eqn (27), the integral along L vanishes now. Hence, using (26) and (5), this yields

$$(2b_1\omega_2 + 2b_2\omega_1)/H = \mathrm{i}\mathbf{Dev}\sigma_0. \tag{30}$$

Finally, it follows from (28) and (30) that

$$b_1 = \omega_1(\mathbf{Dev}\sigma_0 + 2c_1 d_0)\,; \quad b_2 = \omega_2(\mathbf{Dev}\sigma_0 - 2c_1 d_0). \tag{31}$$

Before closing this section we shall draw on eqn (17) to find the average value v_0 of the criterion $v(z)$ over the cell. Making use of the result (21), one can write in the adopted notations

$$\begin{aligned} v_0 &= \frac{1}{H}\int_{S_1} |\psi_1'(z)|^2\,\mathrm{d}x\,\mathrm{d}y + \frac{1}{H}\int_{S_2} |\psi_2'(z)|^2\,\mathrm{d}x\,\mathrm{d}y \\ &= \frac{1}{H}\int_{S_1}\left(\frac{\partial P_1}{\partial x}\right)^2 + \left(\frac{\partial P_1}{\partial y}\right)^2 \mathrm{d}x\,\mathrm{d}y + \frac{1}{H}\int_{S_1}\left(\frac{\partial P_2}{\partial x}\right)^2 + \left(\frac{\partial P_2}{\partial y}\right)^2 \mathrm{d}x\,\mathrm{d}y. \end{aligned} \tag{32}$$

By the Green formula (Mikhlin, 1965), the last expression may be reworked, in view of eqns (31), (25), (23)–(24) and (4), as follows

$$\begin{aligned} v_0 &= \frac{b_1 b_2}{\mathrm{i}H} + \frac{d_0^2}{H}\int_L x\,\mathrm{d}y = \frac{\omega_1\omega_2}{\mathrm{i}H}(\mathbf{Dev}^2\sigma_0 - 4c_1^2 d_0^2) + \frac{h_1}{H}d_0^2 \\ &= \frac{1}{4}\mathbf{Dev}^2\,\sigma_0 + \frac{c_1 c_2(\kappa_2 - \kappa_1)^2}{4(c_1\kappa_2 + c_2\kappa_1 + 1)^2}\mathbf{Tr}^2\sigma_0. \end{aligned} \tag{33}$$

Hence, v_0 proved to be independent of the inclusion shape. This is not necessarily so in the general case of different shear moduli.

From (32) it also follows that $v(z)$ is sectionally subharmonic in the corresponding domains S_1, S_2 (∇^2 stands for the Laplacian)

$$\nabla^2 v(z) = \nabla^2|\psi_j'(z)|^2 = 2\left(\frac{\partial^2 P_j}{\partial^2 x}\right)^2 + 2\left(\frac{\partial^2 P_j}{\partial^2 y}\right)^2 \geqslant 0\,;$$

$$z = x + \mathrm{i}y \in S_j, \quad j = 1, 2$$

and thus attains its maximum values strictly on the boundary (Alhfors, 1965)

$$\max_{z \in S_j} v(z) = \max_{t \in L} v(t)\,; \quad j = 1, 2. \tag{34}$$

4. NECESSARY CONDITIONS OF OPTIMALITY

Our next goal is that of deducing an exact lower bound on the criterion $v(z)$, i.e., given a fixed c_0 to compute explicitly the value $v_{\min}$ from (1) and to prove its attainability by constructing the corresponding optimal composite. For this purpose let us consider for a moment the boundary problem (11) in the 1-connected region S_1 only. By (21) it becomes, after conjugating,

$$\psi_1(t) = F(t) \equiv \overline{f(t)} - \frac{2a_0}{\kappa_j + 1}\bar{t} \tag{35}$$

where the loading function $f(t)$ from (13) is, as yet, undetermined. It is well known (Muskhelishvili, 1977) that the non-homogeneous boundary-value problem (35) has a unique solution if and only if the free term $F(t)$ satisfies the identity

$$\int_L \frac{F(t)}{t - z}\,\mathrm{d}t \equiv 0\,; \quad z \in S_1.$$

A more straightforward necessary condition will also suffice for present purposes. It can be obtained in the following way.

Differentiating (35) with respect to the arc coordinate s on L and integrating the resulting identity with respect to t along L, we get, in terms of (13),

$$\int_L \frac{\partial \psi_1(t)}{\partial s}\,\mathrm{d}t = \int_L \frac{\partial \bar{t}}{\partial s}\left(\sigma_{nn}(t) - \mathrm{i}\sigma_{n\tau}(t) - \frac{2a_0}{\kappa_j + 1}\right)\mathrm{d}t. \tag{36}$$

Noting that (Muskhelishvili, 1977)

$$\frac{\partial t}{\partial s} = \frac{\partial x}{\partial s} + \mathrm{i}\frac{\partial y}{\partial s} = \left(\frac{\partial \bar{t}}{\partial s}\right)^{-1} = \frac{\partial s}{\partial \bar{t}},$$

eqn (36) can be rewritten as

$$\int_L \frac{\partial \psi_1(t)}{\partial \bar{t}}\,\mathrm{d}s = \int_L \left(\sigma_{nn}(s) - \mathrm{i}\sigma_{n\tau}(s) - \frac{2a_0}{\kappa_j + 1}\right)\mathrm{d}s. \tag{37}$$

Let the function $\psi_1(z) = P_1(x,y) + \mathrm{i}Q_1(x,y)$ assume on the L boundary values satisfying the Hölder conditions. Then, by virtue of its analyticity, we (Muskhelishvili, 1975)

$$\begin{aligned}\frac{\partial \psi_1(t)}{\partial \bar{t}} &= \frac{\partial P_1(x,y)}{\partial x} - \frac{\partial Q_1(x,y)}{\partial y} + \mathrm{i}\left(\frac{\partial P_1(x,y)}{\partial y} + \frac{\partial Q_1(x,y)}{\partial x}\right) \\ &= \lim \frac{\partial \psi_1(z)}{\partial \bar{z}} \equiv 0\,; \quad \bar{z} \to \bar{t} \in L\,; \quad (x, y \in L).\end{aligned} \tag{38}$$

As a result, the integral on the left of eqn (38) vanishes. Hence

$$\langle\sigma_{nn}\rangle = \frac{2a_0}{\kappa_1+1}. \tag{39}$$

The angular brackets $\langle\cdot\rangle$ denote the average of the bracketed function over L.

We have thus shown that the average value of the normal stress on L is independent of the inclusion shape when $\mu_1 = \mu_2$. Reached here with a complex variable technique, this conclusion is not entirely new; it was first observed by Hill (1963).

We are now in a position to derive the lower estimate of the quantities $\langle v^+\rangle$ and $\langle v^-\rangle$. Indeed, gathering together eqns (15)–(17), (23), (25), (39), one immediately obtains

$$\begin{aligned} 4\langle v^+\rangle &= \langle(\sigma_{nn}-\sigma_{\tau\tau}^+)^2\rangle+\langle\sigma_{n\tau}^2\rangle \geqslant \langle(2\sigma_{nn}-I_1^+)^2\rangle \\ &= (2\langle\sigma_{nn}\rangle-I_1^+)^2+\langle\Delta^2\sigma_{nn}\rangle \geqslant 0 \end{aligned} \tag{40}$$

and, analogously,

$$\begin{aligned} 4\langle v^-\rangle &= \langle(\sigma_{nn}-\sigma_{\tau\tau}^-)^2\rangle+\langle\sigma_{n\tau}^2\rangle \geqslant \langle(2\sigma_{nn}-I_1^-)^2\rangle \\ &= (2\langle\sigma_{nn}\rangle-I_1^-)^2+\langle\Delta^2\sigma_{nn}\rangle \geqslant \left(\frac{4a_0}{\kappa_1+1}-\frac{4a_0}{\kappa_2+1}\right)^2 = 4d_0^2. \end{aligned} \tag{41}$$

Here $\Delta\sigma_{nn} \equiv \sigma_{nn}-\langle\sigma_{nn}\rangle$, $\langle\Delta\sigma_{nn}\rangle = 0$.

Finally, the following two inequalities hold, on account of relation (34)

$$\max_{z\in S_1} v(z) = \max_{t\in L} v(t) \geqslant 0 \tag{42}$$

$$\max_{z\in S_2} v(z) = \max_{t\in L} v(t) \geqslant d_0^2. \tag{43}$$

Deduced by dropping two terms $\langle\sigma_{n\tau}^2\rangle$ and $\langle\Delta^2\sigma_{nn}\rangle$, the foregoing estimates may appear at the first glance too rough to be attainable. But they both vanish when the function $\psi_1(z)$ is everywhere zero, so that eqn (25) possesses the form

$$\psi_2(t) = d_0\bar{t};\quad t\in L. \tag{44}$$

We combine (42) and (43) in assertion that the maximum values of $v(z)$ over S_1, S_2 are both minimal together under the single optimal condition (44). Coupled with the translation identities (26), this condition presents the external inverse problem of finding a smooth close contour on which a holomorphic function takes the boundary values prescribed by (44). However, as remarked above, we arrive at the same equations in solving the allied problem of elastic energy optimization. The following results obtained in this context are of importance for current purposes.

(1) The existence of a solution can be proved for a certain range of the parameters involved. Generally speaking, the solution exists if the deviatoric part of the external field is sufficiently small when compared to the global part. This restriction may be obtained in a number of ways (Vigdergauz, 1994; Grabovsky and Kohn, 1995). As applied to the case at hand, it takes the form

$$\left|\frac{\mathbf{Dev}\,\sigma_0}{\mathbf{Tr}\,\sigma_0}\right| = \left|\frac{\sigma_{yy}^0-\sigma_{xx}^0}{\sigma_{yy}^0+\sigma_{xx}^0}\right| \leqslant c_2\frac{\kappa_2-\kappa_1}{c_1\kappa_2+c_2\kappa_1+1}. \tag{45}$$

(2) If inequality (45) is met, the equi-stress contours actually exist and admit of various parametric representations (Cherepanov, 1974; Grabovsky and Kohn, 1995; Vigdergauz, 1996) involving the integrals of elliptic functions.

Then, combining (23) and (24), with (42) and (43), the smallest attainable value for a maximum of the local criterion (2) over the cell may be written as

$$v_{\min} = d_0^2 = \frac{(\kappa_2 - \kappa_1)^2}{4(c_1\kappa_2 + c_2\kappa_1 + 1)^2} \mathbf{Tr}^2\, \sigma_0. \tag{46}$$

This maximum occurs simultaneously at each point on the outside of the equi-stress boundary. Here, the stress field anisotropy is completely forced out of the optimal inclusion which remains in the state of all-round constant compression/extension.

It is intriguing that the extremum of the *deviatoric* stresses does not depend on a *deviatoric* part of the average tensor σ_0, whereas the shapes of the equi-stress inclusions do (Cherepanov, 1974; Grabovsky and Kohn, 1995; Vigdergauz, 1996). Put otherwise, all of the optimal structures allowed by inequality (45) are identical in the local strength factor (46). In closing we mention that instead of (2), the von Mizes criterion can be considered quite analogously, because it is derived from $v(z)$ by adding a constant term $I_1(z)$.

5. STRESS DISTRIBUTION AT CIRCULAR ELASTIC INCLUSIONS

Contrary to the general case of different shear moduli μ_1, μ_2 discussed (see the papers cited above) in the context of the energy-wise optimization, the relations (21), (26) and (31) obtained here as well as subharmonicity of the deviatoric criterion $v(z)$ do not constitute conditions of optimality. They result from nothing but the simplifying restriction $\mu_1 = \mu_2$ and thus hold for any smooth inclusion contours. The only difference between the direct and the inverse problem of elasticity is in identity (25) which can be used in two opposite ways. Setting $\psi_1(z) \equiv 0$ in S_1 leads to the desired equation (44) for the contours sought. Alternatively, for a given boundary L one can find the stresses at any point of the cell by solving the same equation (25) with $\psi_1(z)$, $\psi_2(z)$ as unknown functions. Thereafter the increase in strength properties from applying optimal boundaries can be estimated by the dimensionless factor q_0

$$q_0 = \max_{t \in L} [v^{\pm}(t)/d_0^2] \tag{47}$$

where limiting values $v^+(t)$, $v^-(t)$ of the criterion (2) on each side of L are compared with the smallest value $v_{\min} = d_0^2$ from (46).

Equations (25) and (26) present the non-homogeneous Hilbert problem for sectionally holomorphic functions (Muskhelishvili, 1977). With the Weierstrass ζ-function as a quasi-periodic analogue of Cauchy kernel $1/(t-z)$ this problem may be solved in a closed form using line singular integrals over L

$$\psi_j(z) = d_1 z - \frac{d_0}{2\pi i} \int_L \bar{t}\zeta(t-z)\,\mathrm{d}t; \quad z \in S_j + L; \quad j = 1, 2. \tag{48}$$

The off-integral item in the right-hand side of (48) is a general solution (regular in the whole z-plane) to the corresponding homogeneous problem. Coupling the Plemelj formulae (Muskhelishvili, 1977) for the difference in limiting values of a Cauchy integral with the Laurent series expansion of $\zeta(z)$ in a neighborhood of the origin (Abramowitz and Stegun, 1965)

$$\zeta(z) = z^{-1} - \sum_{k=2}^{\infty} c_k z^{2k-1}(2k-1) \tag{49}$$

it can be shown that the boundary condition (25) is really obeyed by (48).

To find the constant d_1, the quasi-periodicity of $\psi_2(z)$ still remains to be used. Substituting eqn (48) into (31) gives, in view of the identity (5)

$$2d_1\omega_1 + 2d_0\frac{h_1}{\pi}\eta_1 = (\mathbf{Dev}\,\sigma_0 + 2c_1 d_0)\omega_1$$

$$2d_1\omega_2 + 2d_0\frac{h_1}{\pi}\eta_2 = (\mathbf{Dev}\,\sigma_0 - 2c_1 d_0)\omega_2 \tag{50}$$

where $2\eta_j = [\zeta(z)]_j, j = 1, 2$.

Equations (50) are interdependent. Combining them we obtain the familiar Legendre relation (Alhfors, 1965)

$$2\eta_1\omega_2 - 2\eta_2\omega_1 = \pi\mathrm{i}. \tag{51}$$

Bearing this in mind one can write the constant d_1, with a little manipulation, as

$$2d_1 = \mathbf{Dev}\sigma_0 - 4\mathrm{i}d_0\frac{c_1}{\pi}(\eta_1\omega_2 + \eta_1\omega_2). \tag{52}$$

For some contours the integral in (48) can be found explicitly. To take an example, let L be a circle of radius r. Then it is easy to check that the following relationships are true

$$\bar{t} = r^2 t^{-1}$$

and

$$\frac{1}{2\pi\mathrm{i}}\int_L t^k\,\mathrm{d}t = \begin{cases}1, & k = -1,\\ 0, & k = 0, 1, \pm 2, \pm 3, \ldots\end{cases}.$$

Employing them along with the expansion (49) we perform term-by-term integration in (48) to obtain

$$\psi_1(z) = d_1 z + d_0 r^2\left(\zeta(z) - \frac{1}{z}\right); \quad z \in S_1 + L$$

$$\psi_2(z) = d_1 z + d_0 r^2\zeta(z); \quad z \in S_2 + L. \tag{53}$$

These formulas can be easily foreseen, since by removing the isolated singularity z^{-1} from $\zeta(z)$ the function $\psi_1(z)$ so defined becomes holomorphic in the closed domain S_1.

To be more specific, suppose the composite is subjected to all-round compression ($\sigma^0_{xx} = \sigma^0_{yy}$) and the cell T is a unit square ($2\omega_1 = -2\mathrm{i}\omega_2 = 1$). Then (Abramowitz and Stegun, 1965) it follows from (51) that $\eta_1 = \mathrm{i}\eta_2 = \pi/4\omega_1$. Substituting this into (52) gives $d_1 = 0$. Using the local properties of the doubly periodic function $\rho(z) \equiv -\zeta'(z)$ (Alhfors, 1965) the following inequalities may be deduced from (53)

$$\max_{t\in L}|\psi'_2(t)| = |\psi'_2(r)| > \max_{t\in L}|\psi'_1(t)| = |\psi'_1(r)|$$

$$\min_{t\in L}|\psi'_2(t)| = |\psi'_2(r\sqrt{\mathrm{i}})| > \min_{t\in L}|\psi'_1(t)| = |\psi'_1(r\sqrt{\mathrm{i}})|. \tag{54}$$

In other words, the extremal values of the criterion $v(z)$ occur invariably on the outside of the inclusion. As a result, the ratio (47) takes a particularly simple form involving the parameters $\alpha = c_1/\pi = r^2$ only

$$q_0(\alpha) = \frac{[\psi'_2(\alpha)]^2}{d_0^2} = \alpha^2|\rho(\sqrt{\alpha})|^2 \quad 0 \leqslant \alpha \leqslant = \omega_1^2 = 0.25.$$

The maximum allowable value of α corresponds to the limiting case when adjacent inclusions

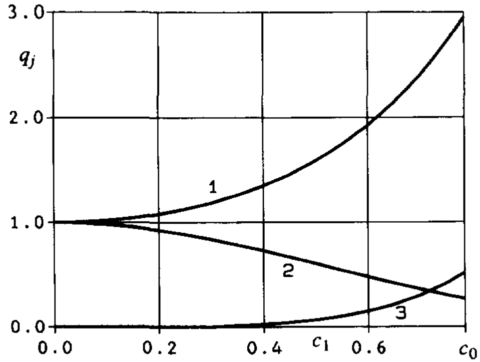

Fig. 2. Composite with circular inclusions. The normalized extreme values of the deviatoric criterion against the volume fraction: the maximum over the cell (1), the minimum over the matrix (2), and the maximum over the inclusion (3).

touch one another. It is interesting that the stress field remains regular at the point of tangency of two circles, because (Abramowitz and Stegun, 1965)

$$q_0(\omega_1^2) = \omega_1^4 |\rho(\omega_1)|^2 = \tfrac{1}{4}\omega_0^4 \approx 2.95426\,;\omega_0 = 1.85407\ldots.$$

The reason is that, by our assumption, the phases are both of the same shear modulus, otherwise the local stresses may increase indefinitely as the circular grains come close together.

In addition to q_0 it is worthwhile to define some other related quantities through formulae (54)

$$q_1(\alpha) = \min_{t\in L}\,[v^-(t)/d_0^2] = \alpha^2 |\rho(\sqrt{\mathrm{i}\alpha})|^2$$

$$q_2(\alpha) = \max_{t\in L}\,[v^+(t)/d_0^2] = \alpha^2 \left|\rho(\sqrt{\alpha}) - \frac{1}{\alpha}\right|^2$$

$$q_3(\alpha) = \min_{t\in L}\,[v^+(t)/d_0^2] = \alpha^2 \left|\rho(\sqrt{\mathrm{i}\alpha}) + \frac{\mathrm{i}}{\alpha}\right|^2.$$

In Fig. 2 we compare q_0, q_1 and q_2 computed as a function of the volume fraction $c_1 = \pi\alpha \leqslant 0.25\pi = c_0$. The factor q_3 is too small to be drawn here at the same scale. From this figure we notice that even the maximum of the deviatoric stresses within the circular inclusion differs markedly from zero in the vicinity of the point c_0 only.

6. CONCLUDING REMARKS

In our opinion the equi-stress conception is one of the most effective methods yet devised to optimize the average characteristics of composites. The lower bounds so derived are of practical significance, since properties of real materials always involve averages over certain domains. Here, the same approach has been successfully applied to the local nonlinear criterion (2); in doing so we have restricted the discussion to a rather specific problem. Nevertheless, the results obtained suggest that the equi-stress inclusions should also be optimal in the general case of different shear moduli $\mu_1 \neq \mu_2$.

It is natural to wonder whether the limitation (46) is adequate for a more real composite in which the inclusions are not exactly the same or/and their spacing is not uniform. This point calls for further investigations. It is apparent, however, that the stress distribution is adversely affected by departures of the structure from periodicity so that the lower bound on the stress concentration remains valid. Combined with the probabilistic assessment technique (Buryachenko and Kreher, 1995 and references therein) it appears to have considerable promise for closer examination of the optimization problem.

Another question that arises is how to expand the results obtained to the 3D case where the theory of a complex variable is no longer applicable. Some promising results along this line are obtained by Wheeler and Kunin (1982), Vigdergauz (1983) and Eldiwany and Wheeler (1986).

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